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# FREE SUBGROUPS OF SMALL CANCELLATION GROUPS

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ABSTRACT

In this paper we give a simple proof of Collin's theorem concerning free subgroups of $C(4)$, $T(4)$ groups. Our proof actually shows that a slender $T(4)$ presentation $\langle x_1, x_2, \ldots, x_n ; r\rangle$ has a free subgroup of rank 2 provided there is a subset $\{a, b, c\}$ of $\{x_1, x_2, \ldots, x_n\}$ with the property that any non-empty freely reduced word in $a$, $b$, $c$ equal to 1 in $G$ has a subword of length 2 contained in an element of $r^*$.

## §1. Introduction

Collins [1] investigated the free subgroup of groups with presentations satisfying the $C(4)$, $T(4)$ conditions (see also Johnson [3]). He has shown that such a group $G$ contains a free subgroup of rank two, except in some cases, which he lists explicitly. The exceptions are all two generator groups.

In [2] the $T(4)$ condition was investigated graphically. Here we give a simple proof, in the spirit of [2], that if $G$ is a $C(4)$, $T(4)$-group which can be generated by three or more elements, then $G$ contains a free subgroup of rank 2. In fact, we prove a slightly stronger result which will be stated explicitly later.

We will say that a presentation $\langle x; r\rangle$ is *slender* if each element of $r$ is non-empty and cyclically reduced, and if for each $R \in r$, no cyclic permutation of $R$ or $R^{-1}$, except $R$ itself, belongs to $r$. We denote by $r^*$ the set of all cyclic permutations of elements of $r$ and their inverses. A word $W$ is called a *piece* (relative to $r$) if there are distinct elements $wu$, $wv$ of $r^*$. The presentation satisfies $C(p)$ ($p$ a positive integer) if no element of $r^*$ is the product of less than $p$ pieces. The *star graph* of the presentation $\langle x; r\rangle$ has vertex set $x \cup x^{-1}$, and $\{x^\varepsilon, y^\delta\}$ $(x, y \in x, |\varepsilon| = |\delta| = 1$ is an edge if and only if $y^{-\delta}x^\varepsilon$ is a subword of some element of $r^*$. The presentation satisfies $T(4)$ if and only if its star graph

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has no triangles (see [2] for details). Two slender presentations $\langle x_1, x_2, \ldots, x_n ; r\rangle$, $\langle x_1, x_2, \ldots, x_n ; s\rangle$ are said to be *equivalent* if there is a permutation $\alpha$ of $\{1, 2, \ldots, n\}$ such that $\theta(r)^* = s^*$ where $\theta$ is the automorphism of the free group on $x_1, x_2, \ldots, x_n$ defined by $x_i \mapsto x_{\alpha(i)}$.

THEOREM. *Let $\langle x_i, x_2, \ldots, x_n ; r\rangle$ $(n \geqq 3)$ be a slender $C(4)$, $T(4)$ presentation not equivalent to a presentation $\langle x_1, x_2, \ldots, x_n ; x_1Q_1, x_2Q_2, \ldots, x_{n-2}Q_{n-2}, s\rangle$ where $s \subseteq r$ and where each element of $\{Q_1, Q_2, \ldots, Q_{n-2}\} \cup s$ involves only $x_{n-1}$, $x_n$. Then the group defined by the presentation has a free subgroup of rank 2.*

REMARK. Our proof actually shows that a slender $T(4)$ presentation $\langle x_1, x_2, \ldots, x_n ; r\rangle$ has a free subgroup of rank 2 provided there is a subset $\{a, b, c\}$ of $\{x_1, x_2, \ldots, x_n\}$ with the property that any non-empty freely reduced word in $a$, $b$, $c$ equal to 1 in $G$ has a subword of length 2 contained in an element of $r^*$. It will be seen (§2) that a presentation satisfying the hypotheses of our theorem has such a subset.

## §2. Preliminaries

Let $\langle x_1, x_2, \ldots, x_n ; r\rangle$ be a slender presentation satisfying the assumptions of the theorem. We want to show that there is a 3-element subset $\{a, b, c\}$ of $\{x_1, x_2, \ldots, x_n\}$ with the property:

*Any non-empty freely reduced word in $a$, $b$, $c$ equal to 1 in $G$*
(†) *has a subword of length 2 contained in an element of $r^*$.*

LEMMA. *A 3-element subset $\{a, b, c\}$ of $\{x_1, x_2, \ldots, x_n\}$ satisfies (†) except possibly if there is an element $R \in r$ where one of $a$, $b$, $c$ occurs exactly once in $R$ and in no other relator.*

PROOF. Suppose there is a non-empty freely reduced word $W$ in $a$, $b$, $c$ equal to 1 in $G$ but which does not have a subword of length 2 contained in $r^*$. Let $M$ be a reduced Van Kampen diagram with boundary label $W$. By standard small cancellation theory (see [4] chapter V) $M$ has a boundary region $\Delta$ (labelled by $R$ say) such that $\Delta$ has at most two interior edges, $\partial\Delta \cap \partial M$ is a consecutive part of $M$, the label $u$ on $\partial\Delta \cap \partial M$ is a subword of $W$. By assumption $u$ has length 1. Since the label on the interior edges of $\Delta$ are pieces, the $C(4)$ condition implies that $u$ is not a piece. Thus $u$ cannot occur in any relator except $R$. Also it occurs once in $R$ because it cannot be part of a label of an interior edge of $\Delta$.

Now suppose $\{x_1, x_2, x_3\}$ does not satisfy (†). Then by the Lemma we can assume (relabelling if necessary) that $x_1$ occurs once in some relator $R_1$ and in no other relator. Some cyclic permutation of $R_1^{\pm}$ will then have the form $x_1 Q_1$ where $x_1$ does not occur in $Q_1$. Now consider $\{x_2, x_3, x_4\}$, and so on. If we do not eventually find a subset satisfying (†) then we will have that $\langle x_1, x_2, \dots, x_n ; r\rangle$ is equivalent to a presentation

$$\langle x_1, x_2, \dots, x_n ; x_1 Q_1, x_2 Q_2, \dots, x_{n-2} Q_{n-2}, s\rangle$$

where $s \subseteq r$ and the elements of $\{Q_1, Q_2, \dots, Q_{n-2}\} \cup s$ involve only $x_{n-1}$, $x_n$ which contradicts the assumption.

## §3. Proof of the Theorem

Let $\{a, b, c\}$ be a subset of $\{x_1, x_2, \dots, x_n\}$ satisfying (†) and consider the full subgraph $\Gamma(a, b, c)$ with the star graph of the presentation $\langle x_1, x_2, \dots, x_n ; r\rangle$ on the vertices $a$, $a^{-1}$, $b$, $b^{-1}$, $c$, $c^{-1}$. Let $\Omega_3$ be the subgroup of the symmetric group on $\{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$ generated by the elements $(xy)(x^{-1}y^{-1})$, $(xx^{-1})$ $(x, y \in \{a, b, c\})$. Then it is shown in [2] that, up to permutating the vertices by an element of $\Omega_3$, $\Gamma(a, b, c)$ is a subgraph of one of the following:

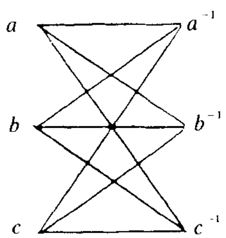


Fig. 1.

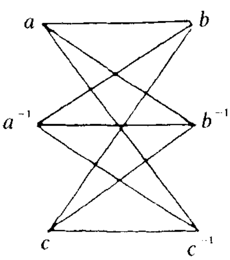


Fig. 2.

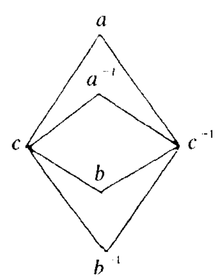


Fig. 3.

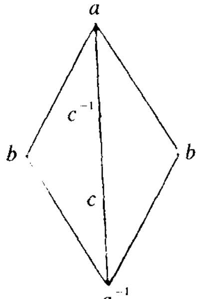


Fig. 4.

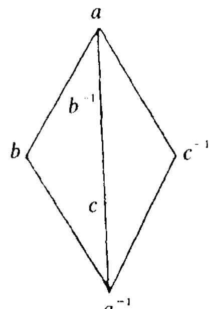


Fig. 5.

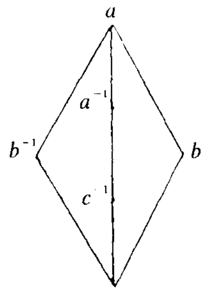


Fig. 6.

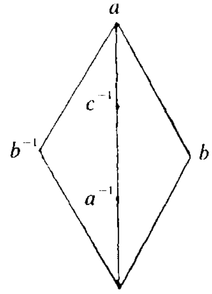


Fig. 7.

It therefore suffices to assume that $\Gamma(a, b, c)$ is a subgraph of one of the above.

Note that a word $xy$ of length 2 ($x, y \in \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$) is a subword of an element of $r^*$ if and only if $\{x^{-1}, y\}$ is an edge of $\Gamma(a, b, c)$.

Suppose $\Gamma(a, b, c)$ is a subgraph of the first graph. Then none of the words $c^{-1}b$, $b^{-1}a$, $c^{-1}a$, $ba^{-1}$, $ca^{-1}$, $cb^{-1}$ occurs as a subword of an element of $r^*$. It follows that $a^{-1}b$, $a^{-1}c$ freely generate a subgroup of the group $G$ defined by the presentation, for no non-empty freely reduced word in $a^{-1}b$, $a^{-1}c$ has (after freely reducing in terms of $a$, $b$, $c$) a subword of length 2 contained in an element of $r^*$. For the remaining cases similar arguments apply. For each case we list in Table 1 two words $u$, $v$ in $a$, $b$, $c$, and we leave it to the reader to verify that if $W(u, v)$ is a non-empty freely reduced word in $u$, $v$, then after freely reducing $W(u, v)$ in terms of $a$, $b$, $c$, we obtain a non-empty word with no subword of length 2 contained in an element of $r^*$.

TABLE 1

| Fig. | $u$ | $v$ |
|---|---|---|
| 2 | $cbc^{-1}$ | $a$ |
| 3 | $bab^{-1}$ | $a$ |
| 4 | $cbc^{-1}$ | $b$ |
| 5 | $ab^{-1}cba^{-1}$ | $cbc^{-1}$ |
| 6 | $cbc^{-1}$ | $aba^{-1}$ |
| 7 | $aba^{-1}$ | $cbc^{-1}$ |

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